Volume discounting under asymmetric information

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This paper-to-be:

- extends the standard Glosten-Milgrom 1985 microstructure model
- and uses it to address volume discounting in equity markets
Volume discounting is a price cut for large quantity orders of a security

- Due to opacity in the order history
  e.g. corporate bonds
- Due to dealer-broker and dealer-trader relationships
  e.g. the LSE
Volume discounting is puzzling. Why?

Traders value assets and trade them for

- Liquidity
- Speculation
- Dividends
Asymmetric information

Agents who trade for dividends
  ▶ buy only if the NPV is greater than the price
  ▶ sell only if the NPV is less than the price

⇒ agents who trade for dividends reveal their beliefs about the NPV by the very act of proposing or accepting a trade
Agents are reluctant to trade if others have superior knowledge. If they trade with a better-informed agent, they lose money. Informed agents who want to trade a lot probably know a lot.

We expect a volume *premium*. 
Microstructure models

My model extends the Glosten-Milgrom family of microstructure models:

► **Dealer** ("market maker") posts a limit order schedule:
  1. Bids: prices at which the dealer will buy quantities
  2. Asks: prices at which the dealer will sell quantities
     Ask > Bid

► **Agents** order from the market schedule
  1. *Informed* get signals and trade on information
  2. *Uninformed* or "noise" agents trade for liquidity or insurance
The classic microstructure model:

- **Asset** (or project) traded has value $V \in \{0, 1\}$
- $Pr(V = 1) = \gamma = \frac{1}{2}$ prior
- Single, *competitive market maker* posts an asking price at which agents may buy one share $p_a$ and a bid price at which they can sell one share $p_b$
- Agents may buy or sell, $a_t \in \{B, S\}$
- Draw informed agent at $Pr(NF) = \delta$, uninformed $Pr(UN) = 1 - \delta$
  1. *Informed* agents receive a signal related to $V$
  2. *Uninformed* “noise” agents buy or sell randomly
- After the trade, the market maker updates his bid and ask
The essential result: Due to asymmetric information even a competitive dealer lowers his bid below $E[V]$ and raises his ask above $E[V]$.

$$p_a = E[V|a_t = B]$$
$$= Pr(NF) \cdot E[V|NF, B] + Pr(UN) \cdot \gamma$$
$$> E[V] = Pr(V = 1) = \gamma$$

Asymmetric information creates a bid-ask spread.
These models are limited because agents can trade only one unit.

If agents could trade $q \in \mathbb{R}$ the dealer could draw inferences from the size of the trade.

In particular, the dealer may infer that he is dealing with an uninformed agent if the size of the trade is high.

$\Rightarrow$ which would make volume discounts incentive compatible.
Asset value $V \in \{0, 1\}$, prior $Pr(V = 1) = \delta$

Agents receive signals $s_i$

1. Informed: $f_{NF}(s|V)$
   and form a belief which constitutes their type $\theta_i = Pr(V = 1|s_i)$

2. Uninformed: $g_{UN}(\theta)$ simply draw a noise $\theta$ (i.e. their demand)

Market maker posts complete schedule $p(q)$, a function of $q \in \mathbb{R}$

Agents buy the risk-averse expected utility maximizing $\{q, p(q)\}$
Tractable signals

- Let $f_{NF}(s_i | V) \sim \mathcal{N}(V, \sigma^2)$ so that $\theta_i = \frac{\delta f_{NF}(s_i | 1)}{\delta f_{NF}(s_i | 1)+(1-\delta)f_{NF}(s_i | 0)}$

  *Note this creates a distribution for theta, $g_{NF}(\theta)$*

- Assign $g_{UN}(\theta)$ a thicker-tailed distribution in the same family.
Tractable signals

Here is what is “driving the model”
Tractable signals

Here is what is “driving the model”
Discrete version

Before working on an analytic solution, I simulated a discrete version with

$$a_t \in \mathbb{Q}$$

Agents have expected CARA utility (no wealth effects)

$$EU(q, p, \theta) = E_\theta [-e^{-\alpha q(V-p)}] = -\theta e^{-\alpha(q-pq)} - (1 - \theta)e^{\alpha pq}$$

A competitive market maker sets break-even prices

$$p(q) = E[V|q = a_t]$$

EU satisfies the Spence-Mirrlees condition so

$q(\theta), p(\theta)$ monotonically increase in $\theta$

$$\Rightarrow p(q) = E[V|q = a_t] = \int_{\theta(q,p(q))}^{\theta(q+1,p(q))} Pr(V = 1|\tilde{\theta})g(\tilde{\theta})d\tilde{\theta}$$

which can be solved numerically.
### Discrete version results

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Price</th>
<th>Total tariff</th>
</tr>
</thead>
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<tr>
<td>...</td>
<td>.50</td>
<td></td>
</tr>
<tr>
<td>p(4)</td>
<td>.50</td>
<td>$2.00</td>
</tr>
<tr>
<td>p(3)</td>
<td>.50</td>
<td>$1.50</td>
</tr>
<tr>
<td>p(2)</td>
<td>.51</td>
<td>$1.01</td>
</tr>
<tr>
<td>p(1)</td>
<td>.58</td>
<td>$0.58</td>
</tr>
<tr>
<td>p(-1)</td>
<td>.42</td>
<td>-$0.42</td>
</tr>
<tr>
<td>p(-2)</td>
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<td>-$0.96</td>
</tr>
<tr>
<td>p(-3)</td>
<td>.50</td>
<td>-$1.50</td>
</tr>
<tr>
<td>p(-4)</td>
<td>.50</td>
<td>-$2.00</td>
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<tr>
<td>...</td>
<td>.50</td>
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</tbody>
</table>
Now $a_t \in \mathbb{R}$. I define a mechanism design problem such that agents find it incentive compatible to reveal their type $\theta$.

The dealer solves

$$\max_{q(\theta), \tau(\theta)} E_\theta \left[ \tau(\theta) - x(\theta) \cdot \hat{\theta}(\theta) \right]$$

s.t.

$$EU(x(\theta), \tau(\theta), \theta) = V(\theta) \geq EU(x(\theta'), \tau(\theta'), \theta) \quad (IC)$$
$$EU(x(\theta), \tau(\theta), \theta) \geq EU(0, 0, \theta) = -1 \quad (IR)$$

where $\hat{\theta}(\theta) = Pr(V = 1|\theta)$
Continuous version

Cannot use the usual IC trick because EU is not quasilinear. Instead,

\[ V(\theta) = -e^{\alpha \tau(\theta)} \left( \theta e^{-\alpha x(\theta)} + 1 - \theta \right) \]

\[ \ln[-V(\theta)] = \alpha \tau(\theta) + \ln \left[ \theta e^{-\alpha x(\theta)} + 1 - \theta \right] \]

\[ \alpha \tau(\theta) = \ln[-V(\theta)] - \ln \left[ \theta e^{-\alpha x(\theta)} + 1 - \theta \right] \]

\[ \ln[-V(\theta)] \] can be obtained by the envelope theorem:

\[ \ln[-V(\theta)] = \int_{\theta_0}^{\theta} \frac{\partial}{\partial \tilde{\theta}} \ln(-V(\tilde{\theta})) d\tilde{\theta} + V(\theta_0) \]

\[ = \int_{\theta_0}^{\theta} \frac{\partial (-V)}{-V} \frac{\partial}{\partial \tilde{\theta}} d\tilde{\theta} + V(\theta_0) \]

\[ = \int_{\theta_0}^{\theta} \frac{e^{-\alpha x(\tilde{\theta})} - 1}{\theta e^{-\alpha x(\tilde{\theta})} + 1 - \tilde{\theta}} d\tilde{\theta} + V(\theta_0) \]
Now the dealer solves

$$\max_{q(\theta)} E_\theta \left[ \int_{\theta_0}^{\theta} \frac{e^{-\alpha x(\tilde{\theta})} - 1}{\theta e^{-\alpha x(\tilde{\theta})} + 1 - \tilde{\theta}} d\tilde{\theta} + V(\theta_0) - x(\theta) \cdot \hat{\theta}(\theta) \right. \\
\left. - \ln \left[ \theta e^{-\alpha x(\theta)} + 1 - \theta \right] \right]$$
Continuous version, step two

Need to integrate this by parts:

\[
E_\theta \left[ \int_{\theta_0}^{\theta} \frac{e^{-\alpha x(\tilde{\theta})} - 1}{\theta e^{-\alpha x(\tilde{\theta})} + 1 - \tilde{\theta}} d\tilde{\theta} \right] = \int_{\theta}^{\bar{\theta}} \int_{\theta_0}^{\theta} \frac{e^{-\alpha x(\tilde{\theta})} - 1}{\theta e^{-\alpha x(\tilde{\theta})} + 1 - \tilde{\theta}} d\tilde{\theta} d\theta
\]

Second trick: Cannot set \( \theta_0 = \theta = 0 \) as usual either. The IC constraints do not bind “upward” locally for the bid contracts but “downward.” Necessary to solve the bid spread “downward” from \( x(\theta_0) = 0 \).

Ask: \[ E_{\theta \in [\theta_0, \bar{\theta}]} \left[ \frac{1 - G[\theta]}{g(\theta)} \frac{e^{-\alpha x(\theta)} - 1}{\theta e^{-\alpha x(\theta)} + 1 - \theta} \right] \]

Bid: \[ E_{\theta \in [\underline{\theta}, \theta_0]} \left[ \frac{-G[\theta]}{g(\theta)} \frac{e^{-\alpha x(\theta)} - 1}{\theta e^{-\alpha x(\theta)} + 1 - \theta} \right] \]

note missing 1
Now the dealer solves

\[
\max_{q(\theta)} E_{\theta} \left[ \frac{e^{-\alpha x(\theta)} - 1}{\theta e^{-\alpha x(\theta)} + 1 - \theta} \frac{\mathbb{I}(\theta \geq \theta_0) - G(\theta)}{g(\theta)} \right] + V(\theta_0) - x(\theta) \cdot \hat{\theta}(\theta) - \ln \left[ \theta e^{-\alpha x(\theta)} + 1 - \theta \right]
\]

FOC:

\[
x(\theta) = \arg \max_x E[\cdot] = f_{nasty}(\theta)
\]

\[
t(\theta) = \frac{e^{-\alpha x(\theta)} - 1}{\theta e^{-\alpha x(\theta)} + 1 - \theta} \frac{\mathbb{I}(\theta \geq \theta_0) - G(\theta)}{g(\theta)} + V(\theta_0) - \ln \left[ \theta e^{-\alpha x(\theta)} + 1 - \theta \right]
\]

Where \( V(\theta_0) = -1 \) by the definition of the outside option not to trade.
$x[\theta]$
Problems

1. \( \frac{t(\theta)}{x(\theta)} \) is tragically not decreasing after some threshold \( \theta \)
Problems

2. If $\gamma = \frac{3}{4}$ or so, the $x(\theta)$ function bends inward

(Or is it a problem? Perhaps a feature?)
3. I am lying to you about $x(\theta)$

I am actually graphing:

$$\max_{q(\theta)} \mathbb{E}_\theta \left[ -\frac{e^{-\alpha x(\theta)} - 1}{\theta e^{-\alpha x(\theta)} + 1 - \theta} \cdot \mathbb{I}(\theta \geq \theta_0) \frac{G(\theta) - G(\theta_0)}{g(\theta)} + V(\theta_0) - x(\theta) \cdot \hat{\theta}(\theta) \right]$$

$$- \ln \left[ \theta e^{-\alpha x(\theta)} + 1 - \theta \right]$$
4. The $t(\theta)$ schedule is not actually symmetric

\[
\begin{align*}
t(.4) &= -0.815124 \\
t(.6) &= 0.839116
\end{align*}
\]
To infinity, and beyond

- Immediate future: Provide a necessary condition for the incentive compatibility of volume discounting
- Later: Use this new model to study the bid-ask spread with opportunistic agents (who can change place in line, who can go twice, etc.)
- Later: Study the rate of convergence of the price to the true value, in particular study whether the market overshoots or undershoots the true price due to the order flow externality. Use this as a basis to study bubble formation (we have no theory of bubble formation, only bubble stability)
- Later: Figure out a way to use the model in an applied setting to recover empirical parameters the traditional G-M model cannot due to its assumption of one-trade-at-a-time